Experimental observation of cnoidal waveform of nonlinear dust acoustic waves

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The experimentally measured waveform of nonlinear dust acoustic waves in a plasma is shown, by analyzing experimental data, to be accurately described by a cnoidal function. This function, which is predicted by nonlinear theory, has broad minima and narrow peaks, and we found that the waveforms in the experimental data match. Fitting the experimental waveforms to the cnoidal function also provides a measure of the wave’s nonlinearity, namely, the elliptical parameter \( k \). By characterizing experimental results at various wave amplitudes, we confirm that the parameter \( k \) varies upward increases and approaches a maximum value of unity, as the wave amplitude is increased. The underlying theory that predicts the cnoidal waveform as an exact solution of a Korteweg-de Vries model equation takes account of the streaming ions that are responsible for the spontaneous excitation of the dust acoustic waves.

I. INTRODUCTION

Waves can easily grow to be nonlinear in dusty plasmas [1–14]. A dusty plasma contains microparticles, i.e., dust grains, which absorb electrons and ions, and thereby gain a large negative charge. When a cloud of these microparticles undergoes a compression or rarefaction, electric fields arise, and this can lead to the propagation of a density wave, which is called the dust acoustic wave (DAW) [15–31]. At low amplitudes, the DAW obeys a linear dispersion relation that has been widely studied; this dispersion relation can depend on physical processes including not only dust particle charge and gas friction, but also other effects such as ion-neutral friction and ion kinetic effects [30]. The amplitude of the DAW can easily attain large amplitudes, due to the large electric forces experienced by the microparticles. Wave electric fields can result in these large forces due to the large charges on each microparticle, so that the microparticle motion associated with the waves can easily become nonlinear. In such nonlinear waves, the dust number density can fluctuate with a large percentage, as can be seen easily in an experiment by video imaging [24, 25]. A common mechanism of exciting the waves in an experiment is energy input from flowing ions [26]. The DAWs have been observed in numerous experiments, for example [16–25, 32] including those under microgravity conditions provided by parabolic flights [33].

Here we demonstrate that the waveform of these nonlinear dust acoustic waves has the shape of a cnoidal function. This has been predicted theoretically for nonlinear waves in various physical systems [34–39]. Theories have also been developed specifically for dusty plasmas [40–46], under various assumptions, predicting cnoidal solutions for nonlinear waves. For example, Yadav et al. [40, 41] derived a theory predicting a cnoidal solution for dust acoustic waves under an assumption that included Boltzmann electrons and ions and cold dust with fluctuating charges. Saini et al [45] used a model consisting of two components of superthermal electrons to study cnoidal waves for nonlinear dust ion-acoustic waves while Tolba et al [46] studied cnoidal forms of dust acoustic waves in positively charged dusty plasmas. The basic nonlinear model equation, of which the cnoidal form is an exact solution, is the classical Korteweg de Vries (KdV) equation that has its historical origins in the study of surface water waves. The KdV equation provides a robust description of finite amplitude low frequency dispersive waves in a wide variety of physical systems and the various assumptions that go into deriving the equation show up as modifications in the coefficients of the equation. We employ an appropriate form of the KdV equation that takes account of the hydrodynamic flow of the ions, and use its cnoidal form solution to match the experimental data.

The experimental literature for cnoidal waveform is more sparse than for the theory. We are aware of only one previous plasma experiment that tested the cnoidal solution; that experiment was for nonlinear drift waves [36] in a non-dusty plasma. Other physical systems that have been studied experimentally, for the cnoidal shape of their nonlinear waves, include surface gravity waves in shallow water [34, 35, 39], and laser interference fringes in the photorefractive bismuth titanate oxide (BTO) crystal [38].

In this paper, we find that the cnoidal wave solution
accurately describes the waveform of nonlinear dust acoustic waves in a ground-based experiment. We will analyze the data from a previous experiment [25], where the level of the wave’s nonlinearity was regulated by adjusting the damping level due to gas drag. In the present analysis, we will fit the experimental data to a cnoidal wave solution of an appropriate KdV equation, derived in (VI). This fit will also yield a useful parameter \( k \) to quantify the nonlinearity of the waves. As a measure of nonlinearity, we will also compare the cnoidal parameter \( k \) to the total harmonic distortion, which was reported in the previous experiment in Ref. [24]. In addition, this will validate an analytical representation of the experimentally observed dust-acoustic fluctuations which will be useful in studying the physics of these fluctuations with an external modulation as envisioned in the PK4 experiments.

II. THEORETICAL FORMULA

The cnoidal wave solution of the KdV equation can be represented in the form,

\[
\phi(x, t) = \beta_2 + (\beta_3 - \beta_2) \text{cn}^2 \left[ 2K(k) \left( \frac{x}{\lambda} + ft \right); k \right], \quad (1)
\]

where, \( \beta_2 \) and \( \beta_3 \) are wave’s minimum and maximum amplitudes, respectively; the function \( \text{cn} \) is one of the Jacobi elliptic functions, with an elliptic parameter \( k = \sqrt{(\beta_3 - \beta_2)/(\beta_3 - \beta_1)} \), where \( \beta_1 \) is a constant; \( K(k) \) is the complete elliptic integral of the first kind of complete elliptic integral; \( \lambda \) and \( f \) are the wavelength and wave frequency, respectively.

In this paper, we will test Eq. (1) with experimental waveform data. This test will also generate a value for the elliptic parameter \( k \) in Eq. (1). The parameter \( k \) characterizes the shape of the cnoidal function. For \( k = 0 \) the cnoidal solution becomes a cosine function, while for values close to unity the cnoidal function gets sharpened peaks and flattened bottoms.

III. EXPERIMENT

Here we review a few key points about the experiment. Further details are found in Ref. [24]. In the experiment, a three-dimensional dust cloud was trapped in an Argon plasma. The plasma was sustained by a radio-frequency (13.6 MHz) voltage applied between a horizontal lower electrode and a grounded vacuum chamber. Ten runs were performed, at gas pressures ranging from 372 to 420 mTorr. Polymer microspheres of 4.8 \( \mu m \) diameter were introduced into the plasma, and they were electrically confined by natural electric fields, which were enhanced in the horizontal direction by a glass box that was open on the top. The fields’ vertical component provided levitation of the particles and it also drove a downward ion flow that could excite DAWs.

The dust cloud was imaged with a digital video camera viewing from the side. Here, we will analyze the image data from the experiment, which were recorded at a speed of 500 frames/s. In Fig. 1, we show snapshots of the dust cloud, which exhibits compressive wave fronts, which are horizontal and propagate. As discussed in Ref. [24], the image intensity is linearly related to the number density of dust particles, due to the design of the experiment.

The degree of the wave’s nonlinearity in the experiment was regulated by adjusting the gas pressure. The gas pressure was different in each of the ten experimental runs. The wave was self-excited by ion flow, and this energy input to the wave competed with gas frictional damping. As seen in the snapshots of Fig. 1(f), for the gas pressure of \( p = 420 \) mTorr, no wave was detectable. By reducing the pressure slightly to 416 mTorr, Fig. 1(e), waves were excited; the wave amplitude was significantly high and even attained a nonlinear amplitude. Further small reductions in the gas pressure yielded even higher wave amplitudes, Fig. 1(a)-1(d). The gas pressure can be used to calculate a damping rate, in s\(^{-1}\), using the formula [6],

\[
\nu_E = 2333 \frac{\delta_E}{p \rho r_d} \sqrt{\frac{2}{Z_{gas} T_{gas}}}, \quad (2)
\]

where the Epstein constant \( \delta_E \) is in the range 1.0 to 1.442. In Eq. (2), \( \rho \) is dust particle’s mass density in kg/m\(^3\), \( r_d \) is particle radius in microns, \( p \) is gas pressure in mTorr, \( T_{gas} \) is gas temperature in K, and \( Z_{gas} \) is the atomic mass of gas molecule. Here, we use \( \delta_E = 1.26 \), as in Ref. [6], \( \rho = 1510 \) kg/m\(^3\) for the particle’s mass density, \( Z_{gas} = 39.948 \) for Argon, and \( T_{gas} = 290 \) K because the experiment was performed with a vacuum chamber at room temperature.

IV. ANALYSIS METHOD

A. Obtaining experimental waveform

Our analysis mainly centers on the waveforms of dust number density fluctuations, which we obtain from a sequence of video images. We exploit the linear relationship of image brightness and dust number density to calculate the density, in arbitrary units.

For an image from one video frame, we choose a region of interest (ROI), as identified in Fig. 2(a). Within the rectangular boundary of the ROI, \( 1.65 \times 0.26 \) mm, we spatially average the image intensity, yielding one instantaneous measure of the number density. We repeat this measurement for every frame of the recorded video...
FIG. 1. Images of a cross section of a 3D dust cloud. Waves are seen in (a) to (e), as indicated by density compression and rarefaction (i.e., the spatial variation in pixel brightness). The observed wave propagated from the top downward, and the wave grew in amplitude as it propagated downward through the cloud. The series of images shown here, (a)-(f), are for six experiment runs, each for a different value of the damping rate, as controlled by gas pressure. Each panel is from one frame of a video.

FIG. 2. (a) Regions of interest ROI-1 to ROI-5. These equally-spaced rectangles (1.65 by 0.26 mm) each span the image’s width. Within a ROI, for one video frame, we spatially average the pixel intensities, yielding a measure of local number density at a particular time, in arbitrary units. Repeating for a sequence of video frames yields a times series of local number density for an experimental run, as in (b) and (c), which are for ROI-3.
FIG. 3. Experimental waveform and cnoidal fit. The experimental data (circles) shown here are smoothed waveforms, normalized by the minimum $n_{\text{min}}$ of the waveform, for ROI-3. Different panels (a)-(e) correspond to different damping levels, as characterized by the damping rate $\nu_E$. The experimental waveforms are nearly symmetrical. In (a) - (c), there is an indication of skewness, which is barely detectable. The fit is shown as continuous curves, computed using Eq. (3). The excellent agreement between the fitted and experimental data demonstrates that the cnoidal wave solution can accurately describe the waveforms of nonlinear dust acoustic waves. The values of $k$, $f$, and $\nu_E$ are presented in Table I.

B. Determining wave’s amplitude and frequency

We obtain the wave’s amplitude and frequency by an inspection of the peaks in the smoothed waveform. The average of the peak values yields the maximum amplitude $\beta_3$ as defined in Eq. (1), while the average of the time interval between the peaks yields the frequency $f$. The minimum amplitude $\beta_2$ is obtained by finding the minimum of the waveform. We then calculate the peak-to-peak amplitude, $H = (\beta_3 - \beta_2)/\beta_2$, which is normalized by the minimum amplitude $\beta_2$.

V. EXPERIMENTAL RESULTS

A. Fitting

Our main result is a fit of the cnoidal function to the smoothed experimental waveforms, in Fig. 3. The data points are the smoothed experimental waveforms, for various damping levels. The continuous curve is

$$\phi_1(x, t) = 1 + H cn^2 [2K(k)ft; k],$$

which is Eq. (1) evaluated at a specific value of $x$. Using only one free parameter, $k$, we fit our experimental data to Eq. (3) by minimizing chi-squared. The fitted curves match the experimental waveforms very well. This agreement was found not only for Fig. 3, (for ROI-3 and five gas pressures), but also for all the data we tested. This agreement is seen for waves with small and large amplitudes alike. At large amplitude (i.e., at low pressure), the features of sharpened peaks and flattened bottoms are captured well by the cnoidal wave solution, Eq. (3).
TABLE I. Experimental parameters and results, for various gas pressure \( p \). The peak-to-peak amplitude \( \Delta n \) is presented two ways, \( \Delta n/n_{av} \) and \( H = \Delta n/n_{min} \), normalized by the mean \( n_{av} \) and the minimum \( n_{min} \), respectively. The damping rate \( \nu_E \), normalized by the frequency \( f \), is calculated using Eq. (2) for each gas pressure. As measures of the wave’s nonlinearity, both the total harmonic distortion THD and the cnoidal fit parameter \( k \) are presented.

<table>
<thead>
<tr>
<th>Gas pressure ( p ) (mtorr)</th>
<th>ROI</th>
<th>Analysis of waveform</th>
<th>Gas damping</th>
<th>Nonlinearity</th>
<th>cnoidal fit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \Delta n/n_{av} )</td>
<td>( H ) (Hz)</td>
<td>( \nu_E/2\pi f )</td>
<td>THD (%)</td>
</tr>
<tr>
<td>372</td>
<td>3</td>
<td>0.88 1.20 27.1</td>
<td>0.66</td>
<td>48.0</td>
<td>0.989</td>
</tr>
<tr>
<td>380</td>
<td>3</td>
<td>0.98 2.34 26.6</td>
<td>0.68</td>
<td>72.7</td>
<td>0.995</td>
</tr>
<tr>
<td>384</td>
<td>3</td>
<td>0.95 1.29 26.1</td>
<td>0.70</td>
<td>55.9</td>
<td>0.993</td>
</tr>
<tr>
<td>392</td>
<td>3</td>
<td>1.37 1.40 25.9</td>
<td>0.73</td>
<td>48.6</td>
<td>0.995</td>
</tr>
<tr>
<td>396</td>
<td>3</td>
<td>1.36 1.41 25.4</td>
<td>0.75</td>
<td>52.5</td>
<td>0.997</td>
</tr>
<tr>
<td>404</td>
<td>3</td>
<td>1.29 1.16 25.3</td>
<td>0.76</td>
<td>81.7</td>
<td>0.997</td>
</tr>
<tr>
<td>408</td>
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<td>0.27 0.30 24.6</td>
<td>0.79</td>
<td>32.2</td>
<td>0.890</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.29 0.33 24.4</td>
<td>0.80</td>
<td>25.5</td>
<td>0.857</td>
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<tr>
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<td>0.59 0.72 24.4</td>
<td>0.80</td>
<td>37.8</td>
<td>0.985</td>
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<tr>
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<td>0.79 1.02 24.6</td>
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<td>53.2</td>
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<tr>
<td>412</td>
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<td>0.99 1.39 24.4</td>
<td>0.80</td>
<td>48.4</td>
<td>0.995</td>
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<tr>
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<td>0.14 0.15 28.2</td>
<td>0.70</td>
<td>10.0</td>
<td>0.708</td>
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<tr>
<td>4</td>
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<td>0.21 0.27 28.7</td>
<td>0.69</td>
<td>32.1</td>
<td>0.863</td>
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<td>0.44 0.53 28.5</td>
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<tr>
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<td>74.2</td>
<td>0.988</td>
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<td>0.69</td>
<td>67.5</td>
<td>0.938</td>
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</table>

This agreement leads us to conclude that the cnoidal wave solution can accurately describe the waveform of nonlinear dust acoustic waves in experimental data. Having thus confirmed the usefulness of the cnoidal solution, we can now turn our attention to practical uses for it. In particular, we will evaluate the parameter \( k \) in the cnoidal formula, as a useful measure of wave’s nonlinearity.

**B. Measures of nonlinearity**

Beyond previous experimental measures of a wave’s nonlinearity, such as the percentage of density fluctuation or the total harmonic distortion THD, we find here that the elliptic parameter \( k \) is also useful for quantifying the wave’s nonlinearity. This parameter is obtained experimentally as the free parameter in the fitting.

As expected, we find that \( k \) increases with increasing nonlinearity, as can be seen in Fig. 3(c) [WHY Figure 3(c) and not just 3?]. For the largest wave amplitudes, the nonlinear waveform has sharpened peaks and a flattened bottom, and in that case we find that \( k \) has a large value of 0.997, which is close to the theoretical maximum of unity. On the other hand, for small wave amplitudes in Fig. 3(e), the waveform is almost sinusoidal, and the parameter \( k \) has a smaller value 0.88. In Fig. 4(a) the parameter \( k \) is shown as a function of wave amplitude.

For comparison, we also calculate the total harmonic distortion (THD),

\[
THD = \sqrt{\frac{A_2^2 + A_3^2}{A_1^2}}. \tag{4}
\]

Here, \( A_1, A_2, \) and \( A_3 \) are the amplitudes of the fundamental, second, and third harmonics of the waves; they are obtained from a Fourier transformation of raw waveforms. The parameter THD was previously used to characterize the nonlinearity of DAWs by Flanagan and Goree [24]. We do the same here, except that our formula Eq. (4) is a ratio of amplitudes instead of powers. Our results for THD are presented in Table I and Fig. 4(b). We find that, as measures of nonlinearity, both \( k \) and THD exhibit monotonic trends, varying upward with wave amplitude, as the wave amplitude is increased from a low level. However, the sensitivity of these parameters is not the same. The parameter \( k \) exhibits most of its variation for relatively low amplitudes \( \Delta n/n_{av} < 0.4 \), as defined in Table I. For larger amplitudes, \( k \) reaches its theoretical maximum of unity. On the other hand, THD exhibits a monotonic trend that does not rapidly saturate at high wave amplitude. This comparison suggests that, as a quantitative indicator of wave’s nonlinearity, the cnoidal elliptic parameter \( k \) has great sensitivity to the nonlinearity at relatively small wave amplitudes, while THD is more useful at larger amplitudes.
FIG. 4. Measures of the wave’s nonlinearity. The nonlinearity is quantified by (a) the elliptic parameter \( k \), and (b) the total harmonic distortion THD. The peak-to-peak amplitude \( \Delta n \) is normalized by a time-average density \( n_{av} \), not by the undisturbed dust density \( n_0 \). Data shown here are taken from Table I. Each data point corresponds to one pressure and one ROI. (For each symbol there is more than one data point because we used various values of the gas pressure.) Note that the parameter \( k \) in (a) reaches its theoretical maximum of unity and no longer varies as wave amplitude \( \Delta n/n_{av} \) increases, while THD in (b) exhibits a monotonic trend that persists even at a higher wave amplitude.

VI. THEORETICAL DISCUSSION

Next we will present a theoretical derivation to further demonstrate that the cnoidal solution of dust acoustic waves is robust. In our derivation, we will make an assumption, including a hydrodynamic flow of ions, that is different from that in previous derivations [40, 41, 45, 46], but we still obtain a solution that is similar to Eq. (2). In this way, we can show that the cnoidal solution is not sensitive to some underlying theoretical assumptions especially those for ions.

In this section we briefly outline a derivation of the model KdV equation appropriate for our experimental conditions. We begin with a simple description of the dusty plasma system as being composed of three fluid components representing the dust, the ions and the electrons [15]. The dust dynamics is then given by,

\[
\frac{\partial n}{\partial t} + \frac{\partial (n v)}{\partial x} = 0 \tag{5}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{Z_d e}{M_d} \frac{\partial \phi}{\partial x} = 0 \tag{6}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = -4 \pi e (n_i - n_e - Z_d n) \tag{7}
\]

Here (5) is the dust continuity equation with \( n \) and \( u \) denoting the dust density and velocity respectively, (6) is the dust momentum equation with \( \phi \) representing the electrostatic potential and (7) is the Poisson equation. Dust pressure is neglected (cold dust fluid). Since the electron mass is negligible compared to that of the massive dust particles it is appropriate to assume the electrons to obey a Boltzmann distribution.

\[
n_e = n_e \exp \left( \frac{e \phi}{T_e} \right) \tag{8}
\]

A similar assumption can also be made for the ions and has traditionally been made in many past derivations [15, 40]. However in the present situation the fast streaming of the ions plays an important role and is responsible for a linear instability that gives rise to the spontaneous excitation of the DAWs. It is appropriate then to take account of the inertial contribution of the ions due to their flow. This can be simply done by adopting the following ‘ballistic response model’ for the ions,

\[
n_i v_i = n_i \rho_i \rho_{i0} \tag{9}
\]

\[
\frac{m_i v_i^2}{2} + e \phi = \frac{m_i \rho_{i0}^2}{2} \tag{10}
\]

where \( v_i, n_i, m_i \) are the ion velocity, ion density and ion mass respectively. \( v_{i0} \) is the equilibrium ion streaming velocity. Quasi-neutrality at equilibrium gives,

\[
n_i = n_{i0} + Z_d n_0 \tag{11}
\]

A linear perturbation analysis of the above equations (5-11), with perturbations \( \sim e^{i(kx - \omega t)} \), gives a linear dispersion relation,

\[
\omega^2 = \beta^2 C_s^2 k^2 \left[ 1 + \frac{k^2 \lambda_D^2}{1 - \eta \theta} \right]^{-1} \approx \beta^2 C_s^2 k^2 \quad \text{for} \quad k^2 \lambda_D^2 << 1 \tag{12}
\]
where $\beta^2 = Z_d(\delta - 1)/(1 - \eta \delta)$, $C_s = (T_e/m_d)^{1/2}$, 
$\delta = n_i0/n_d0$, $\eta = T_e/(m_1 e^2 n_0)$ and $\lambda_D = (T_e/(4\pi n_0 e^2))^{1/2}$.

For finite amplitude waves in the weakly nonlinear limit one can carry out a standard perturbation analysis (e.g. using the reduction perturbation technique using the stretched variables,

$$\xi = \epsilon(x - u_{ph} t); \quad \tau = \epsilon^3 t$$

At the lowest order of the perturbation the analysis yields the linear dispersion relation (and the linear phase velocity) given by (12). Retaining terms of the next significant order gives the following nonlinear evolution equation for the perturbed potential (or density or velocity),

$$\frac{1}{u_{ph}} \frac{\partial \phi_1}{\partial \tau} + \frac{1}{2} \left[ \frac{\delta \eta^2 - 1}{1 - \eta \delta} - \frac{Z_d}{\beta^2} \right] \frac{\partial \phi_1}{\partial \xi} + \frac{\lambda_D^2}{2Z_d} \frac{\delta - 1}{1 - \eta \delta} \frac{\partial^3 \phi_1}{\partial \xi^3} = 0$$

(13)

where $u_{ph} = \beta C_s$ is the linear phase velocity. Eq.(13 has the form of the KdV equation, a fully integrable nonlinear PDE, which has many exact analytic solutions including the well known soliton solutions. An exact periodic solution of the equation (representing a chain of solitons) is the so-called cnoidal wave solution which for (13) is of the form given by Eqn.(1).

VIII. ACKNOWLEDGMENTS

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VII. CONCLUSION

We find that the cnoidal wave solution of Eq. (1) accurately describes the waveform of nonlinear dust acoustic waves in a ground-based experiment. This finding is based on our analysis of experimental waveforms of dust particle’s number density, which are fitted well by the cnoidal solution over a wide range of amplitudes.

The fit yields a useful measure for the wave’s nonlinearity, the so-called elliptic parameter $k$. We find that the parameter $k$ is useful for waves at smaller amplitude. For larger amplitude, the total harmonic distortion THD is found to be a more useful indicator.

We believe it will be interesting to further test the cnoidal wave solution, using the data in other dusty plasma experiments, including PK-4 instrument on the International Space Station [47]. Such experiments, under microgravity conditions, do not require a large vertical electric field, thus have a lesser ion flow.

COAUTHORS: do you wish to add a further conclusion?
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