Collisional plasma sheath model

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The effects of ion collisionality on the plasma sheath are revealed by a two-fluid model. In contrast to previous work, the ion–neutral collision cross section is modeled using a power law dependence on ion energy. Exact numerical solutions of the model are used to determine the collisional dependence of the sheath width and the ion impact energy at the wall. Approximate analytical solutions appropriate for the collisionless and collisionally dominated regimes are derived. These approximate solutions are used to find the amount of collisionality at the center of the transition regime separating the collisionless and collisional regimes. For the constant ion mean-free-path case, the center of the transition regime for the sheath width is at a sheath width of five mean-free paths. The center of the transition regime for the ion impact energy is at a sheath width of about one-half of a mean-free path.

I. INTRODUCTION

When a plasma is in contact with a negatively biased surface, such as an electrode or wall, a strong, localized electric field appears between the plasma and that surface. This ion rich boundary layer, called the sheath,[1] confines electrons in, and expels ions from, the plasma. The energy that ions gain as they fall through the sheath regulates both the physical and chemical processes that occur at surfaces contacting the plasma. Such plasma–surface interactions are important, for example, in plasma processing. Ion collisions in the sheath can significantly reduce the ion impact energy on the surface, and so it is worthwhile to include them in a sheath model.

When collisional effects are considered, three regimes of sheath behavior are evident. There is a collisionally dominated (i.e., mobility limited) regime, a collisionless regime, and a transition regime that separates them. For the collisionally dominated regime, expressions that describe the sheath are available for both the cases of constant ion mean-free path,[2] and constant ion mobility.[3] In the opposite limit, when ion collisionality is negligible, Child's law[4,5] gives a simple description of the sheath. Separating these two regimes is a transition regime that defies simple analytic treatment.

Several authors have recently considered the effects of ion collisionality on the sheath. Jurgensen and Shaqfeh[6] developed a kinetic model for ions suffering charge exchange collisions. Godyak and Sternberg[7] presented a fluid model where the ions experience a collisional drag. However, to the best of our knowledge, the amount of collisionality needed to cause the transition from the collisionless to the collisionally dominated regime has not been treated explicitly. The usefulness of all collisional sheath models has been hampered by the lack of this important information. In this paper we derive simple criteria for the amount of ion collisionality needed for the valid use of the limiting expressions.

We consider a planar, cathodic, source-free, unmagnetized sheath, which is described using a two-fluid model. This model is similar to that used by Godyak and Sternberg,[7] although we allow the ion collision cross section to have a more general (power law) dependence on ion energy. Both the case of constant ion mean-free path and constant ion mobility are contained in this collision model. We solve this model two ways: numerically to yield exact solutions valid over the entire range of collisionality and analytically to give approximate results valid in the collisionless and collisional regimes. We then use these expressions to determine the degree of collisionality at the center of the transition regime.

In the Sec. II we present the fluid equations that are used to describe the sheath. In Sec. III numerical calculations of the sheath thickness and ion energy at wall impact are presented. These useful results reveal that collisions reduce the ion impact energy before decreasing the sheath width. The numerical results motivate the rest of the paper. In Sec. IV we derive solutions of the model in the collisionless and collisional regimes. Using these solutions, analytic expressions for the amount of collisionality at the center of the transition regime based on sheath width and ion impact energy are derived in Sec. V. These are useful for determining whether a given sheath is in the collisionless, collisional, or transition regimes. The main results are summarized in Sec. VI. In the Appendix, we derive an analytic expression for the ion energy lost in the sheath due to collisions when collisionality is small but non-negligible.

II. SHEATH MODEL

A. Governing equations

We consider an unmagnetized, charge-neutral plasma in contact with a planar wall, as sketched in Fig. 1. In the plasma both the density of electrons n_e and the density of (positive) ions n_i are equal to the plasma density n_0. The potential in the sheath is ϕ, and the wall is held at a negative potential ϕ_w. (Throughout this paper, the subscript w will be used to refer to values at this wall.) Consequently, a sheath forms to separate the plasma from the wall. Ions enter the sheath as a cold beam with a velocity v_0 and strike the wall with a velocity v_w, and a kinetic energy $\frac{1}{2}Mv_w^2$. Ions experience a collisional drag inside the sheath. The boundary between the plasma and the sheath is at $x = 0$, and the sheath thickness is D. That is, the wall is at $x = D$. The sheath is assumed to be source-free. (All the symbols used in this paper are tabulated in Table I.) Given these assumptions, our goal is to frame a self-consistent formulation of the colli-

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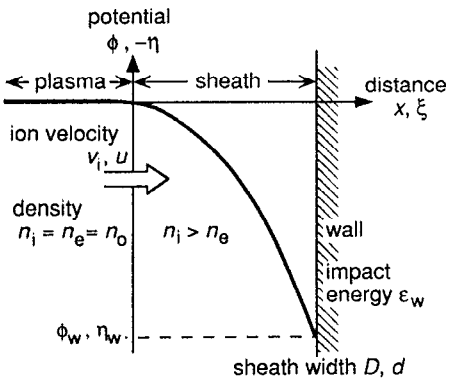

FIG. 1. Model system for the sheath. The potential ϕ is sketched as a function of the distance x from the wall. Dimensionless quantities are shown to the right of their dimensional counterparts for most variables. Ions enter the sheath from the left as a cold beam with a velocity v_0 and strike the wall with a dimensionless impact energy ϵ_w. Ions lose energy in the sheath via ion–neutral collisions.

sional sheath problem and solve it.

We consider governing equations based on a two-fluid model. The electrons are thermalized so their density obeys the Boltzmann relation,

$$n_e = n_0 \exp(e\phi/k_B T_e), \tag{1}$$

where e is the elementary charge, k_B is Boltzmann's constant, and T_e is the electron temperature. The cold ions obey the source-free, steady-state equation of continuity,

$$\nabla\cdot(n_i \mathbf{v}_i) = 0, \tag{2}$$

and motion,

$$M(\mathbf{v}_i\cdot\nabla)\mathbf{v}_i = -e\nabla\phi - \mathbf{F}_c, \tag{3}$$

where the velocity and mass of the ions are $\mathbf{v}_i$ and M, respectively. As the ion fluid travels through the sheath it experiences a drag force

$$\mathbf{F}_c = M(n_n \sigma v_i)\mathbf{v}_i, \tag{4}$$

where n_n is the neutral gas density and σ is the momentum

TABLE I. Symbols used in this paper. The dimensionless sheath theory depends on four parameters: the ion speed at sheath entry u_0, the collision parameter α, the collision exponent γ, and the wall potential η_w. Here $\gamma = 0$ gives the case of constant ion mean-free path, and $\gamma = -1$ gives the case of constant ion mobility. The subscript w is used to refer to sheath variables at the wall.

Dimensional	Nondimensional	Definition
Basic constants		
e		elementary charge
k_B		Boltzmann's constant
ϵ_0		permittivity constant
Plasma parameters		
c_s		ion acoustic velocity $[\sqrt{(kT_e/M)}]$
M		ion mass
n_0		plasma density
T_e		electron temperature
v_0	u_0	ion speed at sheath entry ($u_0 = v_0/c_s$)
λ_D		Debye length in the plasma $\{\sqrt{[\epsilon_0 k_B T_e/(n_0 e^2)]}\}$
Sheath variables		
v_i	u	ion speed [Eq. (8c)]
$\mathbf{v}_i$		ion velocity
x	ξ	distance into the sheath [Eq. (8b)]
	ϵ	ion energy [Eq. (8d)]
ϕ	η	electric potential [Eq. (8a)]
n_e		electron density
n_i		ion density
Collision parameters		
$\mathbf{F}_c$		collisional drag force [Eq. (4)]
n_n		neutral gas density
	α	collision parameter [Eq. (9)]
	γ	collision component [Eq. (6)]
λ_{mfp}		mean-free path for ions $[1/(n_n\sigma_s)]$
σ		ion–neutral collision cross section [Eq. (6)]
σ_s		cross section at the ion acoustic speed [Eq. (6)]
At the wall		
D	d	sheath width ($d = D/\lambda_D$)
v_w	u_w	ion speed at the wall $[u_w = v_w/c_s]$
	ϵ_{max}	maximum ion energy at wall impact [Eq. (13)]
	ϵ_w	ion energy at wall impact [Eq. (8e)]
ϕ_w	η_w	electric potential

transfer cross section for collisions between ions and neutrals. Elastic and charge-exchange collisions contribute to this cross section, which depends on the ion speed v_i. Finally, Poisson's equation relates the electron and ion densities to the self-consistent potential:

$$\epsilon_0 \nabla^2 \phi = -e(n_i - n_e), \tag{5}$$

where ϵ_0 is the permittivity constant.

To complete the model we must specify the dependence of the cross section on ion energy. We assume that it has a power law dependence on the ion speed of the form

$$\sigma(v_i) = \sigma_s (v_i/c_s)^\gamma, \tag{6}$$

where $c_s = \sqrt{(k_B T_e/M)}$ is the ion acoustic speed, σ_s is the cross section measured at that speed, and γ is a dimensionless parameter ranging from 0 to -1. This power law scaling contains the two special cases treated in the existing literature: constant ion mean-free path (constant cross section), $\gamma = 0$, and constant ion mobility, $\gamma = -1$. Real cross sections for the ions streaming through the sheath are more closely approximated by the constant mean-free path case.

Combining Eqs. (1)–(6), we find two coupled, differential equations describing the planar plasma sheath:

$$v_i \frac{dv_i}{dx} = -\frac{e}{M}\frac{d\phi}{dx} - n_n \sigma_s \frac{v_i^{2+\gamma}}{c_s^\gamma}, \tag{7a}$$

$$\frac{d^2\phi}{dx^2} = -\frac{en_0}{\epsilon_0}\left[\frac{v_0}{v_i} - \exp\left(\frac{e\phi}{k_B T_e}\right)\right]. \tag{7b}$$

Equation (7a) is the equation of motion for the ion fluid, and Eq. (7b) is Poisson's equation for the electrical potential. The ion density can be calculated from the ion velocity using the equation of continuity [Eq. (2)], and the electron density can be found from the potential using the Boltzmann relation [Eq. (1)]. With the addition of source terms, these equations would describe the entire discharge.[8]

B. Nondimensional variables

The governing equations can be made dimensionless by an appropriate choice of variables.[1] The electric potential ϕ is scaled by the electron temperature,

$$\eta \equiv -e\phi/k_B T_e, \tag{8a}$$

the distance x is scaled by the Debye length $\lambda_D = \sqrt{[\epsilon_0 k_B T_e/(n_0 e^2)]}$,

$$\xi \equiv x/\lambda_D, \tag{8b}$$

and the ion velocity v_i is scaled by the ion acoustic speed,

$$u \equiv v_i/c_s. \tag{8c}$$

Additionally, the ion kinetic energy is made dimensionless by the electron thermal energy,

$$\epsilon \equiv \tfrac{1}{2}(Mv_i^2/k_B T_e) = \tfrac{1}{2}u^2, \tag{8d}$$

so that the dimensionless ion impact energy at the wall is

$$\epsilon_w = \tfrac{1}{2}u_w^2, \tag{8e}$$

where u_w is the dimensionless ion velocity at the wall. The dimensionless sheath width is $d = D/\lambda_D$, and the dimensionless entry velocity (i.e., the Mach number) is $u_0 = v_0/c_s$.

The degree of collisionality in the sheath is parameterized by α, which is given by the number of collisions in a Debye length:

$$\alpha \equiv \lambda_D/\lambda_{mfp} = \lambda_D n_n \sigma_s, \tag{9}$$

where $\lambda_{mfp} = 1/(n_n \sigma_s)$ is the mean free path for ions traveling with the ion acoustic speed. Note that α is proportional to the neutral gas density n_n. The collisionless case, $\alpha = 0$, is the limit of zero gas density. If the gas density is high enough, or the Debye length short enough, so that the ion mean-free path is one Debye length, then $\alpha = 1$. The average number of collisions in the sheath, which will prove to be a useful quantity, is given by $D/\lambda_{mfp} = \alpha d$.

After the dimensionless variables in Eqs. (8a)–(8c) and Eq. (9) are substituted into the governing equations [Eqs. (7)], those equations become

$$uu' = \eta' - \alpha u^{2+\gamma} \tag{10a}$$

and

$$\eta'' = u_0/u - \exp(-\eta), \tag{10b}$$

where the prime denotes differentiation with respect to the spatial coordinate ξ, so that η' is the dimensionless electric field. As before, Eq. (10a) represents the conservation of ion momentum, and Eq. (10b) is Poisson's equation. These two equations, together with appropriate boundary conditions, provide the description of the collisional sheath that is the main concern of this paper.

To solve these equations boundary conditions must be specified. At the wall ($\xi = d$) the boundary condition is $\eta(d) = \eta_w$. At the sheath–plasma boundary ($\xi = 0$) the boundary conditions are $\eta(0) = 0$, $\eta'(0) = 0$, and $u(0) = u_0$. Note that these conditions are only an approximation to the conditions that actually hold at the sheath–plasma interface. (In fact, the location of the sheath–plasma boundary is not well defined.) To find the correct boundary conditions it would be necessary to include source terms and solve the entire discharge problem self-consistently.

Before considering solutions to the governing equations, we define several more quantities. In the absence of collisions ($\alpha = 0$), the equation of motion [Eq. (10a)] can be integrated once to yield a statement of the conservation of ion energy:

$$\tfrac{1}{2}u^2 = \tfrac{1}{2}u_0^2 + \eta. \tag{11}$$

The collisionless ions strike the wall with an impact energy

$$\epsilon_w \equiv \tfrac{1}{2}u_w^2 = \tfrac{1}{2}u_0^2 + \eta_w. \tag{12}$$

Since collisions can only reduce the impact energy, this represents the maximum energy an ion can have at impact on the wall, which we define as

$$\epsilon_{max} \equiv \tfrac{1}{2}u_0^2 + \eta_w. \tag{13}$$

The fractional energy loss for ions at wall impact due to collisions can then be defined as

$$\Delta\epsilon/\epsilon \equiv (\epsilon_{max} - \epsilon_w)/\epsilon_{max}. \tag{14}$$

With these definitions in mind, we proceed to the numerical solutions.

III. NUMERICAL SOLUTIONS

The governing equations [Eqs. (10)] were solved exactly (i.e., without any approximations) for the electric potential $\eta(\xi)$ and ion velocity $u(\xi)$ by integrating them numerically with a Runge–Kutta[9] routine. We have compared these results to those found using a fully implicit method[10] and find no difference. In the collisionally dominated regime, there is a transient in the solution (the ion speed initially decreases due to the large collisional drag) as it adjusts self-consistently to satisfy the real boundary conditions. We discard this transient behavior so that it has a negligible effect on the calculated sheath solutions.

In Figs. 2 and 3 we plot the sheath thickness d and the ion impact energy ϵ_w as functions of the collision parameter α and wall potential η_w. These plots show three regimes of sheath collisionality. For small α, collisions are negligible, and both d and ϵ_w are nearly independent of α. For large α the ion motion is collisionally dominated; both d and ϵ_w decrease and approach power law asymptotes. Between the collisionless and collisional regimes there is a transition regime. For the collisionless and collisionally dominated regimes, approximate analytic expressions for d and ϵ_w can be derived. The transition regime is much more difficult to treat analytically. Consequently, the numerical results in Figs. 2 and 3 are most valuable for their accuracy in the transition regime.

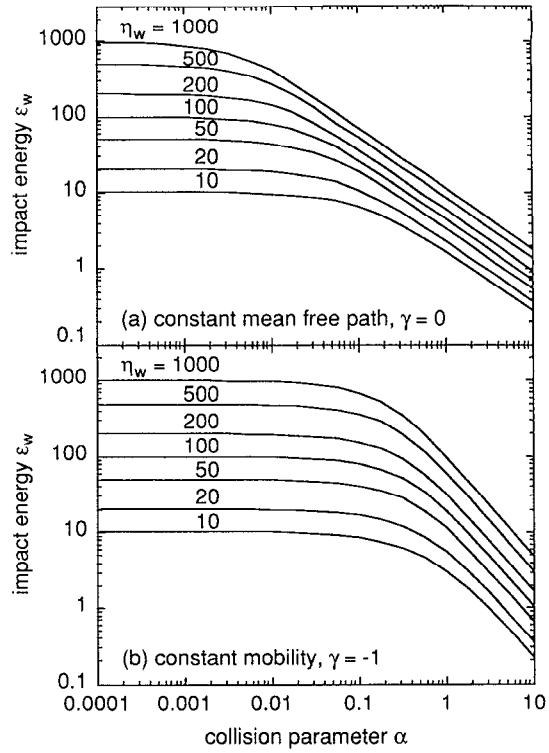

FIG. 3. Exact numerical solution for the average ion energy at impact on the wall ϵ_w as a function of the collision parameter α for various wall potentials η_w. In (a) we show results for the constant mean-free path case, and in (b) for the constant ion mobility case. As in Fig. 2, three regimes are evident. In contrast to the results for sheath width in Fig. 2, the center of the transition regime for ion energy is at smaller values of α (i.e., smaller amounts of collisionality). We have assumed $u_0 = 1$.

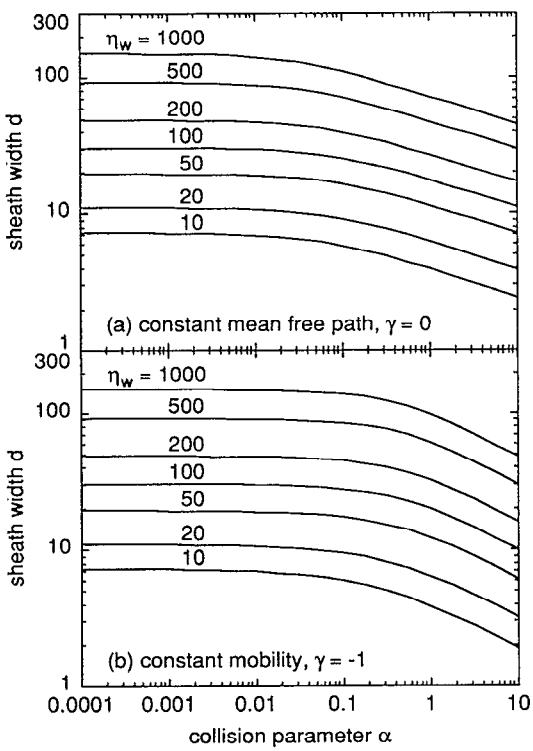

FIG. 2. Exact numerical solutions of the governing equations [Eq. (10)] for the dimensionless sheath thickness d as a function of the collision parameter α for various wall potentials η_w. Here, $\alpha = \lambda_D/\lambda_{mfp}$ is the number of collisions per Debye length λ_D, where λ_{mfp} is the mean-free path for ion momentum transfer. In (a) we show results for constant mean-free path and in (b) the results for constant mobility. Three regimes are evident: a collisionless regime (α small) where d is nearly independent of α, a collisionally dominated regime (α large) where d approaches a limiting asymptote, and a transition regime that separates the collisionless and collisional regimes. We have taken d to be the distance from the wall to the point where $\eta = \ln 2$, and assumed $u_0 = 1$.

When the mean-free path is independent of the ion velocity ($\gamma = 0$) the energy loss is directly proportional to the number of collisions in the sheath, αd. Solutions for the ion impact energy can be made to lie on a universal curve by normalizing the impact energy by the maximum impact energy ϵ_{max} [Eq. (13)] and plotting versus the number of collisions in the sheath, αd. In Fig. 4 we plot the normalized curves. That they all lie on a single curve demonstrates that the energy loss depends only on αd. In Sec. IV C we present an empirical expression for this universal curve.

These exact solutions of our model also provide information about the location of the transition regime and motivate the method we later use to find the center of the transition regime. From Figs. 2 and 3, we see that this regime is centered at a value that is different depending on whether one examines the sheath thickness or the ion impact energy. Accordingly, we use the notation α_d and α_ϵ to describe the center of the transition regime based on the sheath thickness d and ion impact energy ϵ_w, respectively.

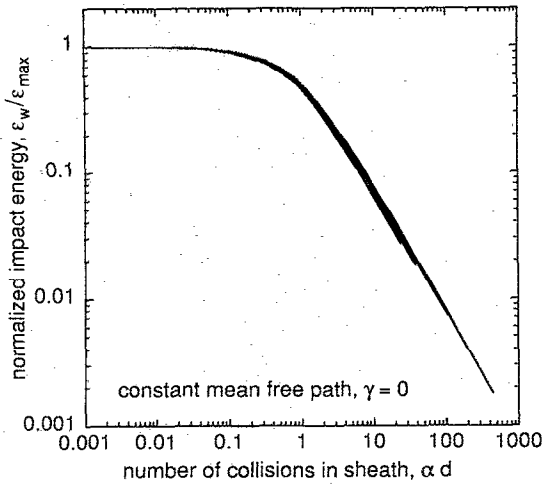

FIG. 4. Normalized impact energy plotted against the number of collisions in the sheath αd for constant mean-free path. [The impact energy data are from Fig. 3(a).] The impact energy has been normalized by the maximum (collisionless) impact energy ϵ_{max}, and the collision parameter has been multiplied by the exact sheath width d [shown in Fig. 2(a)], to give the number of collisions in the sheath. In this special case of constant mean-free path, the impact energy depends only on the number of collisions in the sheath.

IV. APPROXIMATE SOLUTIONS

In this section we derive expressions that give the potential profile, the sheath thickness, and the ion impact energy for collisionless and collisionally dominated sheaths. These results are used in Sec. V to find the center of the transition regime. We give most of our results three ways: for an arbitrary value of γ, for constant mean-free path ($\gamma = 0$), and for constant ion mobility ($\gamma = -1$).

A. Collisionless regime

In the collisionless limit, $\alpha = 0$, we recover Child's law.[4] Inside the sheath, $0 \leqslant \xi \leqslant d$, we seek a power law scaling for the spatial dependence of the electric potential:

$$\eta = a\xi^b. \tag{15}$$

Here $b > 1$ is required, so that the electric field η' will be continuous across the sheath–plasma boundary. When the potential drop across the sheath is large, $\eta_w \equiv -e\phi_w/k_B T_e \gg 1$, we make two approximations.[11] First, the electron term [$\exp(-\eta)$] in Poisson's equation [Eq. (10b)] is neglected. Second, recall that in the absence of collisions energy is conserved [Eq. (11)]. In Eq. (11) the $\frac{1}{2}u_0^2$ term is neglected in comparison to η. To see why this is justified, recall that the Bohm criterion[1]

$$u_0 \geqslant 1 \tag{16}$$

must be satisfied for the collisionless sheath. Typically u_0 is only slightly larger than 1, so that $\frac{1}{2}u_0^2 \ll \eta_w$.

With these two simplifications, the solution of the governing equations is Child's law:

$$\eta = 3^{4/3}2^{-5/3}u_0^{2/3}\xi^{4/3}. \tag{17}$$

By evaluating Eq. (17) at the wall, where $\eta = \eta_w$ and $\xi = d$, we obtain the sheath thickness

$$d = 2^{5/4}3^{-1}u_0^{-1/2}\eta_w^{3/4}. \tag{18}$$

Together, Eqs. (17) and (18) are the familiar Child's law results. It is known that Child's law is not especially accurate,[5] but in this paper we use it for simplicity.

For the collisionless sheath the fractional energy loss is zero, i.e., $\Delta\epsilon/\epsilon = 0$. When there is a small amount of collisionality, we show in the Appendix that the fractional energy loss for constant mean-free path is

$$\Delta\epsilon/\epsilon = \tfrac{8}{9}\alpha d, \tag{19}$$

which depends only on the number of collisions in the sheath, αd. Equation (19) is compared to the exact numerical solution in Fig. 5. For small values of αd agreement is good.

B. Collisional regime

In the limit of strong ion–neutral collisions (i.e., the case of mobility-limited ion motion) the collision parameter α is large. The equation of motion [Eq. (10b)] is simplified under these circumstances by neglecting the convective term on the left-hand side. The resulting equation

$$u^{2+\gamma} = \eta'/\alpha \tag{20}$$

relates the ion velocity to the electric field.[12] This assumption neglects ion inertia and is therefore called a local mobility model.[6]

By inserting Eq. (20) into the Poisson's equation [Eq. (10b)], and neglecting the electron term $\exp(-\eta)$ we again arrive at a power law solution:

$$\eta = \frac{3+\gamma}{5+2\gamma}\left(\frac{3+\gamma}{2+\gamma}u_0\right)^{(2+\gamma)/(3+\gamma)}\alpha^{1/(3+\gamma)}\xi^{(5+2\gamma)/(3+\gamma)}. \tag{21}$$

The collision parameter α appears explicitly in the leading

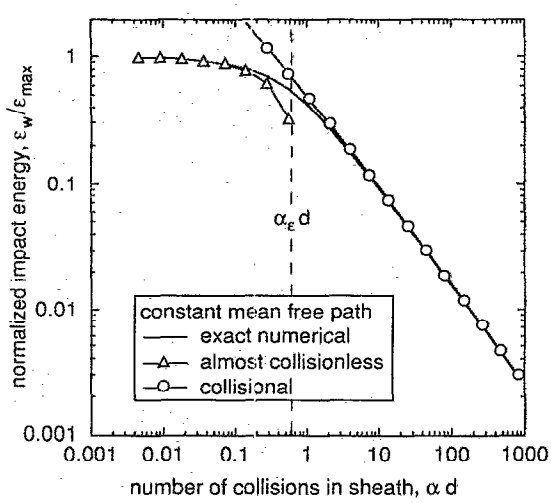

FIG. 5. Comparison between the exact, constant mean-free-path solution for the ion impact energy (shown in Fig. 4) and expressions valid for small and large energy losses due to collisionality in the sheath [Eqs. (19) and (30), respectively]. Both approximations break down in the transition regime. The calculated center of the transition regime ($\alpha_\epsilon d = 0.53$) is also shown.

coefficient, but not in the exponent of ξ. This exponent ranges from $\frac{5}{3}$ for $\gamma = 0$ to $\frac{3}{2}$ for $\gamma = -1$. Both are greater than the value of $\frac{4}{3}$ found in the collisionless regime, giving a narrower sheath. For $\gamma < -2$, this exponent becomes less than 1, so that the electric field η' becomes infinite at $\xi = 0$, and the solution becomes unphysical. This breakdown occurs because the collisional drag decreases as the ion velocity increases.

The sheath thickness, found by invoking the boundary condition $\eta(d) = \eta_w$, is

$$d = \left(\frac{(5+2\gamma)^{3+\gamma}(2+\gamma)^{2+\gamma}}{(3+\gamma)^{5+2\gamma}} \frac{\eta_w^{3+\gamma}}{\alpha u_0^{2+\gamma}} \right)^{1/(5+2\gamma)}. \tag{22}$$

Note that d decreases with increasing collisionality α. This can be understood by considering the governing equations. The viscous drag force reduces the ion velocity in the collisional sheath. To satisfy the conservation of ion flux [Eq. (2)], this smaller ion velocity requires an increase in the ion density. Through Poisson's equation [Eq. (6)], this increase in the ion density leads to a stronger gradient in the electric field, $\nabla^2\phi$. Having a larger gradient means having a smaller scale length, i.e., a smaller sheath thickness d. The decrease in d for increasing collisionality is evident in Fig. 2.

The electric potential η varies not only with ξ and α, but also with the energy dependence of the cross section, characterized by γ. For the special case of constant mean-free path ($\gamma = 0$) the results given above for the electric potential and the sheath thickness simplify to[2,13]

$$\eta = 3^{5/3} 2^{-2/3} 5^{-1} u_0^{2/3} \alpha^{1/3} \xi^{5/3} \tag{23}$$

and

$$d = 5^{3/5} 2^{2/5} 3^{-1} u_0^{-2/5} \alpha^{-1/5} \eta_w^{3/5}. \tag{24}$$

For constant ion mobility ($\gamma = -1$) they simplify to

$$\eta = 2^{3/2} 3^{-1} u_0^{1/2} \alpha^{1/2} \xi^{3/2} \tag{25}$$

and

$$d = 3^{2/3} 2^{-1} u_0^{-1/3} \alpha^{-1/3} \eta_w^{2/3}. \tag{26}$$

We next wish to find the ion impact energy, which can be written using Eq. (20) as

$$\epsilon_w = \tfrac{1}{2} u_w^2 = \tfrac{1}{2} (\eta_w'/\alpha)^{2/(2+\gamma)}. \tag{27}$$

Evaluating η_w' using Eq. (21) we find

$$\epsilon_w = \frac{1}{2} \left(\frac{5+2\gamma}{2+\gamma} \frac{\eta_w u_0}{\alpha^2} \right)^{2/(5+2\gamma)}. \tag{28}$$

Note that the impact energy increases with the sheath potential η_w, but not linearly as it does for the collisionless sheath [Eq. (12)]. For the collisional sheath the fractional ion energy loss defined by Eq. (14) is

$$\frac{\Delta\epsilon}{\epsilon} = 1 - \frac{1}{2} \left(\frac{5+2\gamma}{2+\gamma} \frac{u_0}{\alpha^2 \eta_w^{3/2+\gamma}} \right)^{2/(5+2\gamma)}. \tag{29}$$

For the case of constant mean-free path ($\gamma = 0$), and for $\eta_w \gg 1$, we again find that the impact energy depends on the number of collisions in the sheath αd:

$$\epsilon_w \approx \tfrac{5}{6} (\eta_w/\alpha d), \tag{30}$$

where η_w gives the approximate maximum impact energy and d is the collisional sheath width. This approximate solution for the impact energy [Eq. (30)] is compared to the exact solution in Fig. 5. The fractional energy loss is

$$\Delta\epsilon/\epsilon \approx 1 - \tfrac{5}{6} \alpha d, \tag{31}$$

which should be compared to the energy loss in the almost collisionless sheath [Eq. (19)].

C. Approximate impact energy valid for all α

Because of the importance of the ion impact energy is plasma–surface interactions, we now provide a simple approximate expression for ϵ_w for the case of constant ion mean-free path. As was shown in Sec. III, in this case a single curve describes the collisional dependence of the ion impact energy. Consequently, we can provide a single empirical expression that is valid over the entire range of collisionality. This expression is the ratio of polynomials (i.e., a Padé approximant)

$$\frac{\epsilon_w}{\epsilon_{\max}} \approx \frac{1 + \alpha d}{1 + \frac{13}{7}\alpha d + \frac{6}{5}(\alpha d)^2}. \tag{32}$$

Here the coefficients were chosen to recover the almost collisionless expression [Eq. (19)] for $\alpha d \ll 1$ and the mobility-limited expression [Eq. (30)] for $\alpha d \gg 1$. Agreement with the exact numerical solution is within 2.5% over the entire range of α, with the largest errors in the transition regime.

V. TRANSITION BETWEEN REGIMES

In Sec. IV we developed expressions for the sheath thickness and the ion impact energy appropriate for the collisionless and collisional regimes. However, to correctly use these analytic expressions, one must confirm that the collision parameter α falls in either the collisionless or collisional regime. The exact numerical solutions of the governing equations showed that the transition between the regimes takes place at different values of the collision parameter, α_d and α_ϵ, for the sheath thickness and the average ion impact energy, respectively. Further the location of the transition regime will in general depend on the wall potential and the collision model. In this section we present analytic expressions for α_d and α_ϵ. These expressions are a main result of this paper.

A. Sheath thickness

As seen in Fig. 2, the sheath thickness d approaches a constant limiting value in the collisionless regime, and an asymptote in the collisional regime. We have sketched this situation in Fig. 6. In the collisionless regime, the sheath thickness d is given by Eq. (18). Because this expression is independent of α, this is a constant limiting value, as shown in Fig. 6. In the mobility-limited regime, where the amount of collisionality α is large, d is given by Eq. (22). This asymptote slopes downward.

By definition, the transition regime separates the collisionless and collisional regimes. The asymptote from the collisional regime intersects the collisionless, constant thickness in the transition regime. We take the center of the transition regime α_d to be this intersection point. Accordingly, α_d can be found by equating Eqs. (18) and (22) and solving for α. For an arbitrary value of γ, this procedure yields

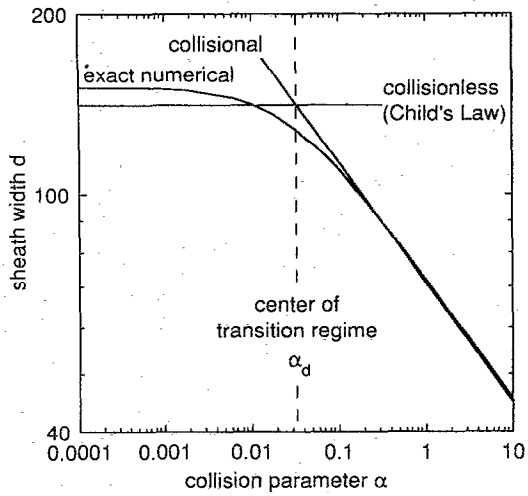

FIG. 6. Sheath width d versus the collision parameter α, illustrating the method we use to find the center of the transition regime. For $\alpha \ll 1$ the sheath width is nearly constant, and is approximated by Child's law. For $\alpha \gg 1$ the ion motion is mobility limited, and the sheath width approaches an asymptote. We use the point where these two approximate expressions intersect to define the center of the transition regime based on sheath width, α_d. The transition value for ion impact energy α_ϵ is similarly defined.

$$\alpha_d = \left[\frac{(5+2\gamma)^{3+\gamma}(2+\gamma)^{2+\gamma}}{(3+\gamma)^{5+2\gamma}}\left(\frac{3}{2^{5/4}}\right)^{5+2\gamma}\right]\frac{u_0^{1/2}}{\eta_w^{3/4+\gamma/2}}. \tag{33}$$

For the constant mean-free-path case this simplifies to

$$\alpha_d = 5^3 2^{-17/4} u_0^{1/2} \eta_w^{-3/4}, \tag{34}$$

so that α_d^{-1} has the same scaling on η_w and u_0 as the collisionless sheath thickness [Eq. (18)]. For the case of constant ion mobility the transition regime is centered about

$$\alpha_d = 3^5 2^{-27/4} u_0^{1/2} \eta_w^{-1/4}. \tag{35}$$

In Fig. 7(a) we have plotted α_d [Eq. (33)] as a function of η_w, and for values of γ ranging from the constant mobility case to the constant mean-free-path case. We see that $\alpha_d \approx 1$, with the exact value depending on the wall potential η_w and on the cross section exponent γ. This means that about one collision per Debye length is required for the sheath thickness to decrease appreciably. The decrease in α_d with increasing wall potential is due to the increase in the sheath width; there are more collisions in a wider sheath.

B. Ion impact energy

Here we perform the same procedure as above, except that we find the center of the transition regime based on the ion impact energy ϵ_w. Examining Fig. 3, the ion energy at impact on the wall ϵ_w approaches a constant value for small α and an asymptotic expression for large α. We take the center of the transition regime based on energy, α_ϵ, to be the point where these limiting expressions cross.

The collisionless value of ϵ_w is given by Eq. (12), while the asymptote for the mobility-limited ion regime is found in Eq. (28). Equating the right-hand sides of these two results to find where they intersect gives

$$\alpha_\epsilon = \left(\frac{5+2\gamma}{2+\gamma}\right)^{1/2} (\eta_w u_0)^{1/2} (2\eta_w + u_0^2)^{-(5+2\gamma)/4}. \tag{36}$$

Since we have assumed $\eta_w \gg 1$, we can simplify Eq. (36) by neglecting u_0^2 compared to $2\eta_w$, which gives

$$\alpha_\epsilon = 2^{-(5+2\gamma)/4}\left(\frac{5+2\gamma}{2+\gamma}\right)^{1/2} \frac{u_0^{1/2}}{\eta_w^{3/4+\gamma/2}}. \tag{37}$$

For the constant mean-free-path case, $\gamma = 0$, this simplifies to

$$\alpha_\epsilon = 5^{1/2} 2^{-7/4} u_0^{1/2} \eta_w^{-3/4}, \tag{38}$$

and for constant mobility we find

$$\alpha_\epsilon = 3^{1/2} 2^{-3/4} u_0^{1/2} \eta_w^{-1/4}. \tag{39}$$

In Fig. 7(b) we exhibit α_ϵ versus wall potential for various values of γ. We see similar behavior to that of α_d except that for the same wall potential $\alpha_d > \alpha_\epsilon$. From Fig. 5 we estimate that the transition regime based on impact energy extends from its center in both directions for roughly an order of magnitude.

These results for the center of the transition regime based on ion impact energy [Eq. (37)] and sheath width [Eq. (33)] have the same scaling with η_w and u_0: only the leading coefficients are different. The ratio of the two transition values is given by a constant that depends only on γ:

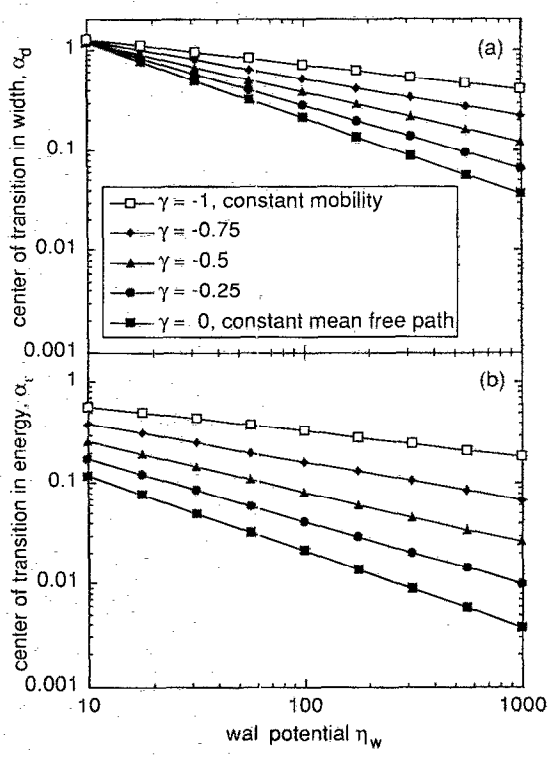

FIG. 7. Values of the collision parameter α about which the transition from the collisionless to collisional regimes is centered. In (a) we plot the center of the transition regime based on impact energy α_ϵ [Eq. (37)], and in (b) the center of the transition regime based on sheath thickness α_d [Eq. (33)]. Dependence of these transition values on the wall potential η_w and the collision cross-section exponent γ are shown. Curves are for $u_0 = 1$.

$$\frac{\alpha_d}{\alpha_\epsilon} = \left(\frac{3}{2}\frac{(5+2\gamma)^{1/2}(2+\gamma)^{1/2}}{3+\gamma}\right)^{5+2\gamma}. \tag{40}$$

For constant mobility ($\gamma = -1$), this ratio has a value of 2.19, which increases to 9.88 for a constant mean-free path ($\gamma = 0$). Consequently, the amount of collisionality needed to reduce the ion impact energy is smaller than that needed to decrease the sheath width.

C. Number of collisions in the sheath at transition

For the constant mean-free-path case, the number of mean-free paths in the sheath, αd, provides a simple criterion for determining whether a sheath is approximately collisionless or collisionally dominated. (For these calculations, we approximate d using the collisionless sheath width.)

The number of mean-free paths in the sheath at the center of the transition regime based on impact energy is

$$\alpha_\epsilon d = \tfrac{1}{3}\sqrt{\tfrac{5}{2}} = 0.53. \tag{41}$$

That is, we expect a significant decrease in impact energy at the wall when the sheath is only $\approx$one-half of a mean-free path wide. For sheath widths much less than one half of a mean-free path, the impact energy is well predicted by the expression for the almost collisionless sheath, as shown in Fig. 5. For widths much greater than one-half of a mean-free path, the mobility limited expression [Eq. (30)] should be used. For sheaths with widths comparable to the transition width, one must use either the exact numerical solutions or the empirical expression given by Eq. (32).

For the transition in width we find that the transition regime is centered on

$$\alpha_d d = \tfrac{1}{3}(\tfrac{5}{2})^3 = 5.21. \tag{42}$$

If the sheath width is significantly less than five mean-free paths then Child's law [Eq. (18)] should be used, and if the width is much greater than five mean-free paths, the mobility limited expression [Eq. (24)] is appropriate. In the transition regime the numerical solutions must be used.

VI. SUMMARY

We have presented a fluid model for the collisional plasma sheath that includes a power law dependence of the ion collision cross section on energy. Special cases of this dependence include both constant ion mean-free path (constant cross section) and constant ion mobility. Approximate solutions of this model appropriate for the collisionally dominated sheath were derived. In particular, we derived expressions for the potential profile, sheath width, and ion impact energy at the wall (see Sec. IV). In addition, an expression for the ion impact energy appropriate in the almost collisionless sheath was presented (see the Appendix), as was an empirical expression for the ion impact energy valid over the entire range of collisionality (Sec. IV C).

By equating the expressions for the collisional sheath width and impact energy to corresponding ones in the collisionless case, we have calculated the location of the center of the transition regime that separates the collisionless and collisional regimes (Sec. V). In the special case of constant ion mean-free path, we find that the sheath width is reduced by collisions when it is about five mean-free paths wide. However, the ion impact energy on the wall is significantly decreased when the sheath width is only approximately one-half of a mean-free path.

ACKNOWLEDGMENT

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APPENDIX: ENERGY LOSS IN THE ALMOST COLLISIONLESS SHEATH

While the ion energy lost in the sheath can be found from a numerical solution of the governing equations, it is sometimes more convenient to have a ready analytic expression. In Sec. IV, we presented a result for $\Delta\epsilon/\epsilon$ in the highly collisional regime, $\alpha \gg \alpha_\epsilon$ [Eq. (30)]. When collisions are neglected entirely, the ions do not lose any energy and $\Delta\epsilon/\epsilon = 0$. In this appendix we improve on the collisionless result by deriving an analytic expression [Eq. (A8)] that is valid for the almost collisionless sheath, i.e., for $\Delta\epsilon/\epsilon \ll 1$. We treat the case of constant ion mean-free path, $\gamma = 0$.

Our approach here exploits the fact that $\alpha_\epsilon < \alpha_d$; therefore, the ion impact energy is reduced by collisions before either the sheath thickness d or the electric potential profile $\eta(\xi)$ is appreciably altered. This means that we can find the ion energy lost by integrating the equation of motion [Eq. (10a)] while using the electric potential and width predicted by the collisionless Child's law [Eqs. (17) and (18)].

Our objective here is to find ϵ_w so that we can find the fractional energy lost $\Delta\epsilon/\epsilon$ [Eq. (14)]. To evaluate the ion velocity at wall impact u_w, we integrate the equation of motion [Eq. (10a)] from the edge of the sheath ($\xi = 0$) to the wall ($\xi = d$):

$$\int_0^d uu'\, d\xi = \int_0^d (\eta' - \alpha u^2) d\xi, \tag{A1}$$

so that

$$\frac{1}{2}(u_w^2 - u_0^2) = \eta_w - \alpha\int_0^d u^2\, d\xi. \tag{A2}$$

The ion impact energy on the wall is

$$\begin{aligned}\epsilon_w &\equiv \tfrac{1}{2}u_w^2 \\ &= \frac{1}{2}u_0^2 + \eta_w - \alpha\int_0^d u^2\, d\xi.\end{aligned} \tag{A3}$$

We now make the crucial approximation that the ion velocity $u(\xi)$ in the integrand of Eq. (A3) is given by the collisionless velocity, which satisfies

$$\tfrac{1}{2}u^2 = \tfrac{1}{2}u_0^2 + \eta, \tag{A4}$$

where η is given by Child's law [Eq. (17)]. This approximation limits the validity of the following results to the case where the fractional energy loss is small. Doing this, we obtain

$$\begin{aligned}\epsilon_w &= \frac{1}{2}u_0^2 + \eta_w - \alpha\int_0^d (u_0^2 + 2\eta) d\xi \\ &= \frac{1}{2}u_0^2(1 - 2\alpha d) + \eta_w - 2\alpha\left(\frac{9u_0}{4\sqrt{2}}\right)^{2/3}\int_0^d \xi^{4/3}\, d\xi.\end{aligned} \tag{A5}$$

Performing the integral in Eq. (A5), and making use of Eq. (18) for the collisionless sheath thickness d, we find

$$\epsilon_w = (\eta_w + \tfrac{1}{2}u_0^2) - \alpha d(u_0^2 + \tfrac{6}{7}\eta_w). \tag{A6}$$

The first term on the right gives the collisionless impact energy, and the second term is the energy loss due to collisional drag. The energy loss is proportional to the number of collisions in the sheath, αd. In the limit $\eta_w \gg u_0$, for which Child's law is valid, the impact energy is

$$\epsilon_w \approx \eta_w (1 - \tfrac{6}{7}\alpha d), \tag{A7}$$

so that fractional energy loss [defined by Eq. (14)] in the almost collisionless sheath is

$$\Delta\epsilon/\epsilon \approx \tfrac{6}{7}\alpha d. \tag{A8}$$

This is the principal result of this appendix. We require $\Delta\epsilon/\epsilon \ll 1$ for this expression to be accurate, or alternatively $\alpha d \ll 1$ (i.e., $\alpha \ll \alpha_\epsilon$). In the mobility-limited regime, where $\alpha \gg \alpha_\epsilon$, one should use instead Eq. (30).

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[12]More accurate calculations made with a kinetic model show that for constant ion mean-free path the drag force is $F_c = (\pi/2)Mv_i^2/\lambda_{mfp}$. See, for example, Ref. 6. This is in contrast to the fluid model used here, which predicts $F_c = Mv_i^2/\lambda_{mfp}$. To use the more accurate kinetic expression for the drag force in the case of constant ion mean-free path, multiply α by $\pi/2$.

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